We examined the codevelopment of mathematical concepts and the mathematical practice of defining within a sixth-grade class investigating space and geometry. Drawing upon existing literature, we present a framework for describing forms of participation in defining, what we term aspects of definitional practice. Analysis of classroom interactions during 16 episodes spanning earlier and later phases of instruction illustrate how student participation in aspects of definitional practice influenced their emerging conceptions of the geometry of shape and form and how emerging conceptions of shape and form provided opportunities to develop and elaborate aspects of definitional practice. Several forms of teacher discourse appeared to support students’ participation and students’ increasing agency over time. These included: (a) requesting that members of the class participate in various aspects of practice, (b) asking questions that serve to expand the mathematical system, (c) modeling participation in aspects of practice, (d) proposing examples that create contest (i.e., monsters), and (e) explicitly stating expectations of and purposes for participating in the practice.

Key words: Concepts in practice; Mathematical definitions; Mathematical practices

In recent years, the field of mathematics education has advocated for an expanded view of what it means to know mathematics and to participate in mathematical activity. Kilpatrick, Swafford, and Findell (2001) advocated a shift away from a primarily procedural view of mathematics to a more encompassing view that includes developing relations between conceptual and procedural forms of mathematics and learning to participate in epistemic practices of knowledge creation and revision, such as defining, making conjectures, and proving. More recently in the United States, the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) present educators with a challenge: to reorient pedagogy toward the coconstitution of mathematical concepts and practices. Such a reorientation requires understanding how students’ participation

The authors would like to thank Leona Schauble, Ilana Horn, and Phil Cooke, as well as the anonymous reviewers, for their insights and thoughtful feedback that helped to shape this work. Earlier versions of this article were presented at the American Educational Research Association Annual Conference in Vancouver, Canada, 2012, and in San Francisco, 2013; at the annual meeting of the Jean Piaget Society in Chicago, Illinois, 2013; and at the International Conference of the Learning Sciences in Sydney, Australia, 2012.
in mathematical practices can be cultivated in ways that allow for the productive generation and development of mathematical concepts. Because students are rarely well versed in disciplinary practices, mathematical practices and conceptual understanding must co-originate (Harel, 2008), in contrast to experts for whom existing well-formed practices inform the development of new disciplinary understandings (Lakatos, 1976; Wilkerson-Jerde & Wilensky, 2011; Wineburg, 1998). To investigate how emerging mathematical practices and concepts could be coconstituted, we conducted a 6-month design study in which middle school students had opportunities to engage in the mathematical practices of definition, conjecture, refutation, inquiry, and proof as they investigated shape and form on the plane (Lehrer, Kobiela, & Weinberg, 2013). In this article, we examine how student participation in the practice of defining influenced their emerging conceptions of the geometry of shape and form and how emerging conceptions of shape and form provided opportunities to exercise and elaborate this form of practice.

Characterizing the Practice of Defining

Lave (1993) suggested that doing and knowing are inherently inventive because learning to do something is a form of improvisation conducted with the experiential and material resources at hand. In some disciplines, forms of activity are endorsed as robust means for producing and refining knowledge; they constitute epistemic practices that are distributed across particular configurations of social and material resources (Knorr Cetina, 1999). Mathematical practices are recurrent forms of activity governed by social norms that serve the purpose of creating and refining knowledge. Epistemic practices and disciplinary concepts are coconstituted, so that a fusion of concepts in practice is appropriate (Hutchins, 2012). That is, “concepts in the wild are manifest in practices” (Hutchins, 2012, p. 315); as practices develop or change, so do concepts. As we will later elaborate, to learn to engage in practice, students must be inducted into these forms of activity. Hence, students’ identification and recognition of practice is emergent and involves gradual appropriation of particular forms of activity.

Defining is a distinctive form of mathematical practice. In contrast to axioms, definitions are contested rather than taken for granted, and unlike lemmas, theorems, or corollaries, definitions cannot be proven. In this sense, mathematical definitions are distinct from other mathematical entities—questions, conjectures, axioms, lemmas, theorems, or corollaries—because they are the negotiated grounds for all mathematical work. As a result, defining is closely coupled with the development and revision of new concepts and ways of understanding. For example, Lakatos (1976) described the historic relations between contested definitions of polyhedron and the genesis of topological understandings of shape.

Mathematical definitions serve several purposes. First, definitions are used to introduce new objects to the field of mathematics (Borasi, 1992; Zaslavsky & Shir, 2005). As objects are introduced and used, definitions are created to describe their essential properties and relations. Similarly, definitions are used to distinguish among objects (Lakatos, 1976) so that participants in a mathematical community...
Marta Kobiela and Richard Lehrer have a practical means for communicating about mathematical ideas (Borasi, 1992). Other mathematical practices are built directly on systems of definition. For example, a procedural definition of angle, as directed rotation, supports a path perspective that makes sensible theorems about the sum of the exterior angles of any polygon (Abelson & diSessa, 1980).

The practice of defining is typically underemphasized in school mathematics; instead, students often experience definitions as received from an authority. However, an emerging body of research suggests that student participation in the generation, revision, and evaluation of definitions is a productive site for developing mathematical concepts, ranging from ideas about spatial properties and objects, such as straightness (Ouvrier-Buffet, 2006), triangle (Lehrer, Jacobson, Kemeny, & Strom, 1999; Zandieh & Rasmussen, 2010), and Platonic polyhedra (Lehrer & Curtis, 2000), to number properties and relations, such as even and odd (Ball, 1993) and the Fibonacci sequence (Leikin & Winicki-Landman, 2001), to algebraic concepts, such as increasing functions (Zaslavsky & Shir, 2005). These studies suggest that opportunities for conceptual development provided by participation in the practice of defining span early elementary to undergraduate years.

Aspects of Definitional Practice

These studies further suggest that the practice of defining is not monolithic but instead involves the coordination of distinct but related forms of activity, which we term aspects of definitional practice. We have identified eight aspects of practice, each observable in multiple studies: (a) proposing a definition, (b) constructing or evaluating examples or nonexamples, (c) describing properties or relations, (d) constructing definitional explanations or arguments, (e) revising definitions, (f) establishing and reasoning about systematic relations, (g) asking definitional questions, and (h) negotiating criteria for judging adequacy or acceptability of definitions. Because practice is about the orchestration of forms of activity, these aspects of definitional practice are intentionally related to one another and often co-occur. At the same time, each aspect highlights a distinct form of definitional activity.

The first aspect of practice, proposing a definition, occurs whenever students are invited to construct a definition (e.g., Larsen & Zandieh, 2005; Zandieh & Rasmussen, 2010). This aspect of practice is often critical for initiating discussions and debate about definitions. Proposing definitions is often aided by constructing or evaluating relevant examples or nonexamples of the object being defined (e.g., Ball, 1993; Herbst, 2005; Keiser, 2000; Lehrer, Randle, & Sancilio, 1989). These examples or nonexamples are mediated typically by concept images of what students perceive a defined object should look like (Vinner, 1983, 1991; Vinner & Dreyfus, 1989).

When proposing definitions and evaluating examples or nonexamples, it is important to have students describe properties or relations of the object being defined that are mathematically relevant (e.g., Ambrose & Kennehan, 2009; de Villiers, 1998; Keiser, 2000; Lehrer et al., 1999; Mariotti & Fischbein, 1997). Because students may not describe properties or relations either when proposing...
a definition or when constructing or evaluating an example or nonexample, we propose that it is necessary to distinguish among these different aspects of practice. For instance, children’s descriptions of geometric objects are often holistic, describing the overall look without referencing properties (e.g., both “look like a mummy house,” Ambrose & Kenehan, 2009, p. 165).

A fourth aspect, constructing definitional explanations or arguments, involves justifying the inclusion or exclusion of a definition, negotiating about the relevance of properties or relations of an object, and arguing for the inclusion or exclusion of an example of a definition (e.g., Herbst, Gonzales, & Macke, 2005; Lehrer & Curtis, 2000; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). Definitional arguments are provoked by competing claims about the validity or utility of a definition or by contest about the status of an example, often assisted by teachers who revoice or otherwise bring into contact disparate student contributions. In contrast to other forms of mathematical argument, such as proof, that rely on definitions, definitional arguments refer to the process of negotiating competing claims related to mathematical definitions. For instance, Lehrer, Jacobson, Kemeny, and Strom (1999) described one child’s argument during a class’s construction of a definition of triangle. The child had constructed a triangle with three paper strips, one of which was a curved strip. When the class rejected her example as a triangle, she disagreed, appealing to their collectively constructed definition of “three sides and three corners” (Lehrer et al., 1999, p. 78). She stated, “No, but it doesn’t matter. Look [gesturing to the board], it has three corners [gesturing to each vertex] and three sides [gesturing to each strip of paper]” (Lehrer et al., 1999, p. 78). Note that the student described the shape’s properties but did so with the purpose of justifying why the example should be classified as a triangle based on the class’s definition.

As illustrated in the previous example, definitional arguments often motivate students to engage in the practice of revising definitions to include additional properties or relations, to eliminate unnecessary or inaccurate properties (e.g., Borasi, 1992; de Villiers, 1998; Lin & Yang, 2002), or to better conform to features or roles that definitions should play (Zaslavsky & Shir, 2005). Historically, revisions often occur when existing definitions are not precise enough to distinguish examples from nonexamples (Lakatos, 1976).

Defining may also involve establishing and reasoning about systematic relations in the mathematical system established by the scope of the definition. Relations may be between general classes of objects or between related properties within classes of objects (e.g., Herbst, 2005; Lehrer et al., 1989; Ouvrier-Buffet, 2006).

Further discussion or investigations of these systematic relations are often provoked when members of the class (e.g., students or teacher) ask definitional questions about components of a definition; about examples of definitions; about the qualities, properties, or relations of the objects being defined; or about the nature of definitions (e.g., Leikin & Winicki-Landman, 2001; Zandieh & Rasmussen, 2010). For instance, in Borasi’s (1992) study, when students were considering the definition of isosceles triangle, one student asked whether the property of “two
equal angles” was sufficient for determining a triangle to be isosceles: “Does it really guarantee that if a triangle has two equal angles then it is isosceles?” (p. 34).

Finally, students may also **negotiate criteria for judging adequacy or acceptability** of definitions more generally. For example, topics of negotiation include what constitutes a definition (Leikin & Winicki-Landman, 2001), whether a definition needs to be economic, or whether any property (e.g., the latent parts, such as the diagonal of a square) may serve as the definition or as part of the definition (Borasi, 1992; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005).

Although the investigators of student-centered definition described above noted other forms of participation, these eight aspects were most prevalent in our study and were often coordinated, such as when proposed definitions provoked further contest about the adequacy of a definition, which in turn instigated further revision. Moreover, as we noted earlier, defining and conceptual development operated in tandem during the course of these investigations. For instance, defining helps broaden students’ initial images of the objects they are investigating, often enabling them to generate or evaluate new examples or nonexamples of objects (Borasi, 1992; Keiser, 2000; Lehrer et al., 1989; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). Defining also appears to support students’ description of objects, moving them away from holistic descriptions toward more mathematical descriptions that focus on relevant parts and properties (Ambrose & Kenehan, 2009; Lehrer et al., 1999). In other cases in which students attended to mathematical features, they learned to reason about which features to include in the definition and which were not relevant to the definition (e.g., Borasi, 1992; de Villiers, 1998; Herbst, et al., 2005; Keiser, 2000; Lehrer & Curtis, 2000; Lehrer et al., 1999; Lin & Yang, 2002; Zaslavsky & Shir, 2005). For example, when defining “perfect” solids, the students in Lehrer and Curtis’s (2000) study continuously revised their conjectured rules as they evaluated new examples and nonexamples. One conjecture, that three faces needed to meet at each vertex, was eliminated once a student noticed that the octahedron had four faces coming together at each vertex. In a few cases, students also improved in writing economic definitions (Borasi, 1992; de Villiers, 1998; Lin & Yang, 2002). Considerations of economy of definition may provoke investigation of relations among relevant features and, hence, what needs to be stated explicitly and what is implied by explicit statement.

**Definition as Accomplished in Interaction**

Although defining is accessible to a wide range of students, supports the growth of conceptual understanding, and involves coordination of multiple aspects of practice, very little is known about how students’ participation in defining develops. In particular, the longer term trajectory of the coconstitution of a system of mathematical concepts and aspects of definitional practice is unclear. To investigate potential trajectories of development, we drew upon theories and methods of discourse analysis in which defining can be considered as a participation framework governed by normative expectations about appropriate forms of participation (Goffman, 1981; O’Connor & Michaels, 1996). Building normative...
understanding of practice requires negotiations between community members in
order to develop taken-as-shared (Yackel & Cobb, 1996) views about what consti-
tutes joint activity, a process that Goffman (1981) described as establishing the
footing of a conversational frame. Such negotiations are made in interaction
among community members (Ellis, 2011; Jurow, 2004). During these interactions,
members of a group take on different roles as they are positioned in terms of
“competence, authority, and accountability” (Greeno, 2005, p. 88), both by them-

• How are aspects of the practice of defining and concept development cocon-
  stituted over time?
• In light of our emphasis on concepts in practice, what forms of interaction
  promote the coordination of aspects of practice with mathematical concepts?

Method

Instructional Context

Participants and setting. The participants \(n = 18\), 10 male) were an ethnically
diverse class of sixth-grade students who attended an urban school serving primarily
underrepresented youth in the southeastern region of the United States. Most
students (75%) qualified for free or reduced lunch. Half of the participants came
from traditional classrooms in which procedural mathematics was emphasized. The
other half of the participants had the same teacher the previous year (for Grade 5).

Instructional design. The students’ work with definitions was situated within a
larger project aimed at engaging students in mathematical inquiry about geometry
and spatial mathematics (Lehrer et al., 2013). The students spent the first week of
the school year working within a Connected Mathematics Project (CMP) unit on
polygons. Our work with the students then began the following week and lasted
through February. During this time, all of the mathematics instruction focused on
topics in geometry and spatial mathematics, except for a single class on number and
operations. This instruction was conducted twice each week for 1.5 hours per class.
One of us (the second author) served as a visiting mathematics instructor during this
time with occasional instruction from the first author and from the classroom
teacher. Thus, the students’ regular mathematics instruction was replaced by the
geometry instructional intervention. Other mathematical topics, such as data, statis-
tics, and probability, were investigated in the remaining months of the school year.

There were a total of 46 geometry lessons during the year. Topics included
definitions of polygons and related properties and objects, interior and turn angle
sums for triangles and then polygons, symmetries of polygons, definition and
properties of rhombi, tiling of polygons, relations between the number of sides of a polygon and the number of diagonals, triangle definition and congruency theorems, the Pythagorean theorem, isometries and symmetries, and area measure of polygons. Given our focus on cultivating a culture of inquiry, students also spent class time discussing qualities of good questions. Figure 1 shows how topics were sequenced and the relative amount of time spent investigating them. Although the bulk of students’ work with geometric definitions was concentrated within the first 7 days of instruction, students visited and revisited definitions throughout the year. As we later describe, students explored fundamental properties of space and relations among these properties as they constructed definitions. Moreover, definition was intertwined with other forms of mathematical activity, such as generating conjectures and theorems about the objects defined, so the time span devoted to definition was longer than might be typical of conventional instruction.

Investigations were generally guided by students’ questions and conjectures rather than a written curriculum. Nonetheless, there were several key features of the instructional design. First, investigation of geometry began with asking students to define the term *polygon* and related properties, such as “side,” “angle,” and “straight.” These definitional investigations comprised the first few weeks of school. This aspect of instruction was consistent with student work in CMP, although CMP did not position students as generators of definitions. Students were given opportunities to construct both structural definitions (e.g., describing properties that constitute an object) and procedural definitions (e.g., describing how to construct an object) (Eylon & Reif, 1984; Zaslavsky & Shir, 2005). Second, mathematical questions were highlighted as important, and starting early on, the teacher asked students to pose questions about particular mathematical objects (such as a square drawn on the board) and documented their questions on a list that was displayed in the classroom. Third, students routinely investigated questions from the collective questions list. At times, investigations were open-ended and students chose their approach, whereas at other times, the teacher guided investigations, for example, by suggesting a set of cases to generate and test. Fourth, because questions and conjectures often led to investigations that led to new questions or conjectures, students were engaged in the creation of a mathematical system of definitions, conjectures, questions, investigations, theorems, and proofs (Lakatos, 1976). As definitions stabilized, students used them in service of other forms of mathematical practice. For instance, on Day 10 of instruction, a student justified the measure of the turn angle of a figure as 90 if the interior angle was 90 “because a straight line is 180-degrees.” Finally, throughout the instruction, the teacher capitalized on students’ everyday experiences of space to help them reason about objects, properties, and relations among them. Thus, many of the opportunities for negotiating and elaborating definitions were generated via accessible bodily experiences (e.g., polygons as paths), an important resource for mathematical learning (Henderson & Taima, 2005; Papert, 1980). For example, while walking, students experienced angles as turns and straight as a constant heading. (For more information about the instructional
Figure 1. The mathematical topics explored by the class. In the figure, “Qs” stands for “questions,” “Def” for “definitions,” “Quads” for “quadrilaterals,” and “Prop” for “properties.”
Marta Kobiela and Richard Lehrer

intervention and the impact on students’ learning, see Lehrer et al., 2013.)

Procedure

Each lesson (except for one) was videotaped and digitally rendered for further analysis. One camera captured whole-group discussion. During small-group work, this camera roamed between the different groups with the intent of providing snapshots of work from a range of students. Sound was captured via microphones suspended in the ceiling of the room. About 4 weeks in, a second camera was mounted onto the wall to capture the interactions of one small group of three students. We took field notes of whole-group interactions in order to supplement the video and serve as a platform for reflection. At the end of each lesson, field notes were compiled, and these served to guide the next lesson. We also collected additional forms of data, including written assessments, student journals, and interviews. However, because our goal was to characterize collective interactions, we used the field notes and video records of whole-group activity and looked at small-group interactions for additional evidence about qualities evident in the whole group.

Analysis

Our analysis consisted of three parts: (a) characterizing participation in aspects of practice, (b) characterizing the mathematical concepts developed by the class, and (c) characterizing how aspects of practice and concepts codeveloped. We begin by describing our sampling procedure for selecting episodes of classroom activity to analyze and then describe each of the three parts of analysis.

Sampling procedure for selecting classroom episodes. Our sampling procedure consisted of three phases of data reduction.

Phase 1. Using video records and field notes, we noted days that included instances when students were engaged in the negotiation of a definition for more than a few turns of talk, signified by competing ideas. This resulted in 21 potential days.

Phase 2. In order to conduct a detailed analysis of participation in aspects of practice, we selected 16 definitional episodes. We defined definitional episodes to be segments of whole-group discussion that involved one or more of the following: (a) the negotiation of a mathematical definition (e.g., when competing or alternate definitions were discussed between at least two members of the class); (b) discussion, including justification, of relations between two or more classes or properties (e.g., the difference between angles and vertices); or (c) discussion, including justification, of relations between a particular case and a class (e.g., whether a sketched rectangle is a regular polygon). Questions that motivated the start of a new definitional episode often included those directed toward classification (e.g., “What is a polygon?”) and those directed toward properties of objects (e.g., “What does straight mean?”). We limited definitional episodes to moments from whole-class discussions because, although small-group activity influenced
whole-group activity, we were mainly interested in how knowledge and aspects of practice became taken-as-shared (Yackel & Cobb, 1996) in particular moments (we later describe how we operationalized taken-as-shared). Definitional episodes were framed around one or two mathematical objects or properties that were under discussion. In cases in which two definitional episodes about the same object (e.g., a polygon) were separated only by small-group work, we documented them as one episode instead of two.

As Figure 1 shows, the 16 selected definitional episodes were from 4 different days (i.e., five episodes from Day 1, four from Day 4, four from Day 6, and three from Day 26). We chose these 16 definitional episodes because they spanned students’ initial entrée into defining (Days 1, 4, and 6) and a later time (Day 26) at which we expected aspects of definitional practice to have stabilized. In addition, the episodes were selected to allow for parallel comparisons. In particular, the chosen episodes (a) were centered on the definition of the same or a similar object, (b) involved investigations of the same form of definition (e.g., structural definitions), and (c) were similar in activity structure. For example, on Days 1, 4, and 6, conversations began with the question “What is a polygon?” Similarly, in the final segment of time (Day 26), the discussion was motivated by the question “What is a triangle?”

All the definitional episodes were transcribed to describe talk, inscriptions, bodily motion, and gesture. Parentheses were used to denote descriptions of inscriptions, gesture, and bodily motion in order to see how these forms of expression highlighted meaning in talk, such as messages about aspects of practice. Moreover, at times, embodied communication existed without talk and reflected how a participant thought about a mathematical idea (e.g., angle-as-rotation, turning one’s body to communicate an amount of turn). We also added additional detail to the transcripts, borrowing conventions from Dressler and Kreuz (2000), to highlight stressed and overlapping talk in order to see what participants positioned as important and also to gauge the level of engagement from students. For forms of stress, we noted elongated syllables (::), emphasized words (CAPS), and indicated rising (⁄) and falling (\) intonations. We used brackets ([ ] ) to indicate overlapping talk, and parentheses (( )) to indicate description of gesture, bodily motion, or action. We then parsed these transcripts into turns of talk, which served as our unit of analysis. We counted a turn of talk to be any uninterrupted contribution, consisting of talk, gesture, or bodily motion. Interrupted contributions counted as multiple turns of talk with the number of turns depending on the number of interruptions. This amounted to 553 turns of talk in total (176 for the episodes from Day 1, 138 for the episodes from Day 4, 110 for the episodes from Day 6, and 129 for the episodes from Day 26).

Phase 3. Finally, we selected five more definitional episodes between Day 6 and Day 26, which we term transitional episodes, to corroborate prospective pathways of development. We selected Day 8 because students spent time constructing definitions of triangle, and this provided a contrast to their work on Day 26. On Day 17, students revisited the definition of polygon and related properties, providing a comparison point to their earlier work. On Day 19, students
began constructing definitions of rhombi. Because a rhombus, like a triangle, is a subclass of polygon, it provided another window to changing aspects of practice and of coordination between aspects of practice and concepts.

**Characterizing participation in aspects of definitional practice.** To characterize how members of the class participated in the aspects of definitional practice, we developed operational descriptions of the eight aspects of practice identified in previous definition studies. We then further refined these initial categories by iteratively coding samples of the data, initially using the 13 definitional episodes of whole-group activity from Days 1, 4, and 6 of mathematics instruction. When coding, we noted turns of talk that did not fit our initial coding scheme, and this led to the clarification or elaboration of categories. Finally, we checked the codes using the remaining three definitional episodes from Day 26 of instruction and made slight revisions to the coding scheme.

Using the finalized coding scheme, the first author coded the entire sample of 553 turns of talk once more. When coding, a turn of talk could receive multiple codes. In instances when an utterance spanned multiple turns of talk (for instance, if the speaker was interrupted), then both turns of talk received the code. For all categories, the contributions did not have to be conventionally mathematically correct in order to be coded. An abbreviated coding scheme is presented in Table 1, and we further elaborate on the categories as we present the results.

**Characterizing mathematical concepts.** Using the talk, gestures, or inscriptions of the members of the class to characterize mathematical concepts, we documented two features of the mathematical ideas explored by the class. Our intention was not to make claims about what individuals were thinking but rather to capture the nature of the mathematical system explored by the class. The mathematical system included all objects, properties, examples, nonexamples, and relations described between those entities, correct or incorrect.

First, for each turn of talk, we noted all mathematical objects and properties mentioned (e.g., polygon, side, angle) and any proposed examples or nonexamples of the objects (e.g., a drawn or gestured square). For instance, in the turn of talk “A polygon has the same angles and the same length of uh, same lengths of sides,” we noted the following mathematical objects (italicized): polygon, angles, and sides.

Second, we noted how members of the class related mathematical objects. This included relations between (a) a class and a subclass (e.g., polygon and regular polygon) or a case of the class (e.g., polygon and a drawn rectangle), (b) a class and the properties that describe that class (e.g., polygon in relation to sides and angles), and (c) a class and another class (e.g., side and angle). In each of these cases, we first noted how members of the class described inclusion or exclusion between the entities (e.g., whether something “is” or “is not” the other or “has or has not” the other). Additionally, we noted descriptions of how classes, subclasses, or properties were or were not related. For example, in the definition “a regular polygon has the same sides,” we noted that same was used to relate regular
Codevelopment of Concepts and Defining

Characterizing the codevelopment of concepts and aspects of definitional practice. We compared the analyses of mathematical concepts and the participation in aspects of practice side by side in order to develop descriptions of how interactions involving definitional practice contributed to the development of a system of mathematical concepts and how the concepts, in turn, informed participation in aspects of practice. To do so, we first looked at points in our analysis of concept development when an object, relation, or property was first introduced during a definitional episode. We compared these instances to what was happening at the same moment with respect to class members’ participation in aspects of practice. Likewise, we identified points of shift in aspects of practice and compared these instances to our analyses of concept development. We termed these moments in which concepts and aspects of practice informed one another points of contact between concepts and practice. For each point of contact, we then characterized the types of interactions that occurred in that moment in order to understand how concepts in practice (Hutchins, 2012) were constituted in particular moments of interaction. We started by looking at how members of the class participated in aspects of practice. For example, if the teacher asked a definitional question, we considered what types of questions he asked. To illustrate, during the first definitional episode, when a student first related “same sides” and “same angles” to “polygon,” we noted that this introduction occurred when this student was prompted by the teacher to propose a potential definition of polygon (“Who can help me understand what a polygon is?”). We also looked for other forms of participation not captured by aspects of practice. For example, we noted that on Day 26, several students articulated a message about participating in defining that the teacher had articulated earlier and that this message helped resolve a conflict in what should be included in the students’ definition of triangle. In all, we identified five types of interactions that seemed especially important in explaining how aspects of definitional practice and mathematical concepts were coconstituted. We elaborate on these five interactions in the results section that follows.

We then looked to see who was contributing to the creation of points of contact in order to identify the roles of the students and teacher and whether those roles shifted over time. Drawing upon Goffman’s (1981) notion of frames, we considered the nature of the interactions, including the aspects of practice and other forms of interaction, to constitute the framings that members of the classroom community took on in relation to the practice. We considered student-generated contributions to be moments when a student introduced an idea (e.g., an object, relation, or property) for the first time during the episode. Student-generated contributions were often motivated by the teachers’ questions or by other students’ contributions. Thus, we were not trying to make claims that these contributions were spontaneous and not prompted by interaction with others. Rather, we were concerned about noting opportunities for students to contribute
<table>
<thead>
<tr>
<th>Aspect of practice</th>
<th>Description</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposing definitions</td>
<td>Constructs a definition for a mathematical object. Proposed definitions do not have to include mathematical properties.</td>
<td>“A polygon has the same angles and the same length of uh, same lengths of sides.”</td>
</tr>
<tr>
<td>Describing properties or relations</td>
<td>Articulates, through talk or writing, properties and relations of a class of mathematical objects or a particular case of a class. Properties and relations may be described in service of goals, such as constructing an explanation or proposing a definition.</td>
<td>“Yeah cause [squares] have 4 angles and 4 sides.”</td>
</tr>
<tr>
<td>Constructing or evaluating examples</td>
<td>Constructs an example of the object being defined or determines whether a particular example belongs to a set. May be in service of constructing definitional arguments or explanations or in service of evaluating a definition.</td>
<td>A student suggests that a polygon is defined as “sides and angles.” The teacher constructs an example using their definition of three connected but not closed lines (a “Z” like figure). A student then evaluates the example: “That’s not a polygon.”</td>
</tr>
<tr>
<td>Constructing definitional explanations and arguments</td>
<td>Justifies a claim about a definition, example of a definition, qualities of an object being defined, or relations between two classes of objects. Definitional arguments are often preceded with qualifiers such as because.</td>
<td>“A circle wouldn’t be a polygon cause a circle doesn’t have sides.”</td>
</tr>
<tr>
<td>Revising definitions</td>
<td>Adds properties to, eliminates properties from, or modifies elements of a definition. May also include reassigning a definition to a new set (see example).</td>
<td>One student claims that a polygon “has the same angles and the same length of uh, same lengths of sides.” Another student notes instead, “all regular polygons,” suggesting that the definition is instead relevant for regular polygons.</td>
</tr>
<tr>
<td>Establishing and reasoning about systematic relations</td>
<td>Establishes, considers, or reasons about relations between two or more general classes of objects or properties OR elaborates a definition of an object that is part of the definition of another object being defined (e.g., elaborates on side because it is part of the definition of polygon).</td>
<td>“A square is a polygon” (establishing the class of squares is inclusive in the class of polygons).</td>
</tr>
<tr>
<td>Asking definitional questions</td>
<td>Asks a question about a definition or about qualities, properties, relations, or examples of the object being defined.</td>
<td>“[When we define polygon] do we need to say sides and angles or is it enough to say sides?”</td>
</tr>
<tr>
<td>Negotiating criteria for judging adequacy or acceptability of definitions</td>
<td>Negotiates with another speaker which features or roles of definitions should be used to determine whether a definition is adequate or acceptable.</td>
<td>One group defined a triangle as three sides, three angles, and closed. A student said their definition needed to include “straight sides.” Two of the group members protested, and in doing so negotiated about the features of a definition: “But we already said sides.” “That’s the definition of sides.”</td>
</tr>
</tbody>
</table>
ideas and for students’ contributions to be the focus of discussion.

We also considered whether mathematical concepts or aspects of practice became taken-as-shared. To determine whether aspects of practice were taken-as-shared, we compared the teacher’s and the students’ engagement in aspects of practice at different points in time. We then noted when students’ engagement resembled that of the teacher’s. To determine whether concepts were taken-as-shared, we noted (a) when a definition was proposed without any contest and (b) when such definitions were then used in service of other definitions. In doing so, our aim was to seek evidence of appropriation of aspects of definitional practice by members of the collective. Finally, we checked our descriptions generated from the selected 16 definitional episodes with the transitional definitional episodes in order to generate confirming or disconfirming evidence of appropriation.

Results

We identified five types of interactions that seemed especially important in explaining how aspects of definitional practice and mathematical concepts were coconstituted: (a) requesting that members of the class participate in various aspects of practice, (b) asking questions that serve to expand the mathematical system, (c) modeling participation in aspects of practice, (d) proposing examples that create contest, and (e) explicitly stating expectations of and purposes for participating in the practice. In what follows, we present the results chronologically in order to illustrate how these five interactions developed and changed in relation to shifts in practice and concepts.

We begin by describing the class’s initial work with definitions during Day 1 to highlight how the teacher initiated students into the practice of defining in ways that provided them immediate access to participate in practice and contribute ideas to the communal mathematical system. In particular, he did so by requesting their participation in aspects of practice and by asking them questions that expanded the mathematical system, all while modeling those interactions. We then show how monsters played a key role in Days 4 and 6 to further expand the class’s mathematical system. Also during Days 4 and 6, students took on increasing authority in expanding the mathematical system, as they appeared to appropriate the forms of participation the teacher had been modeling. Throughout this time, the teacher communicated expectations for how students should participate in the practice of defining. In our final section, we illustrate how the early establishment of those expectations appeared to play a key role in supporting the development of students’ dispositions for participating in practice, as was evident in Day 26.

Initiating Students Into the Practice of Defining

Despite Day 1 being the students’ first collective entrée into defining, they participated in all aspects of definitional practice, although some aspects were more salient than others. Students most frequently reasoned about systematic relations (e.g., a square is a polygon), but also spent considerable time constructing definitional explanations or arguments, evaluating examples, describing properties, and
proposing definitions. Moreover, by participating in aspects of definitional practice, students contributed to developing the mathematical system of concepts. Students proposed potential properties of a polygon, including “same sides,” “same angles,” and “angles,” as well as related classes of objects, “regular polygon,” “octagon,” “circle,” and “square.” Contest about the word regular instigated introduction of properties of “same angles” and “same sides,” later modified as “congruent sides” and “congruent angles” as the teacher probed about the meaning of “same” and asked them to recall more precise language to describe this relation. Except for the introduction of the term polygon and an example of a rectangle, the students introduced all other properties, relations, and objects to the discussion, illustrating the magnitude of student authorship in constructing the mathematical system.

During Day 1, aspects of practice and concept development were coordinated through the teacher’s requests for students to participate in various aspects of definitional practice, such as proposing potential definitions. For example, he began the discussion by asking, “What is a polygon? . . . Okay, give me the most general definition you can. So that I can recognize a polygon and I could tell the difference between a polygon and a turnip.” In response, students introduced objects, properties, or relations that had not been previously mentioned. For instance, Vern proposed: “A polygon has the same angles and the same length of uh, same lengths of sides.” The teacher then revoiced Vern’s proposal, resulting in a revised definition by another student:

\[ T: \text{ Vern’s claim is that all polygons have the SAME length of sides and the SAME angles. Rachel.} \]
\[ R: \text{ All regular polygons.} \]
\[ T: \text{ All REGULAR polygons (pointing at Rachel and looking at Vern). Do you accept her amendment?} \]
\[ V: \text{ yeah} \]
\[ T: \text{ All REGULAR polygons.} \]

By revoicing Vern’s statement and labeling it as a claim, the teacher positioned him as participating in the aspect of practice of proposing definitions. Similarly, the teacher also revoiced Rachel’s contribution and labeled it as an amendment, positioning her as participating in revising definitions. The teacher’s initial verbal stress on the word same indicated potential properties of polygon, and his subsequent emphasis on the word regular indicated the nature of the revision to the definition proposed by Rachel. Figure 2 highlights how, in this example, the teacher’s request for students to propose a definition allowed for the expansion of the class’s mathematical system.

In the example above, the teacher introduced a new object, but in most cases, he instead requested that students propose definitions about objects that they introduced. For example, when Lavona proposed a new definition for polygon, “I
Figure 2. Requesting participation in aspects of definitional practice. The figure illustrates how the teacher’s request for students to propose a definition of polygon invited them to participate in aspects of practice and contribute to the introduction of new objects, properties, or relations. Aspects of practice in which members of the class participated are indicated in the gray rectangles. Ovals represent classes of objects or properties mentioned by members of the class. Lines represent relations between objects and properties or other objects (both correct and incorrect). Labeled edges indicate the nature of the relation between the two properties (e.g., “same” is used to show that the students described polygons as having the same sides and same angles). Bold and underlined transcript is used to indicate mention of the objects, properties, or relations introduced in that frame.

think all shapes are polygons except for . . . uh a quadrilateral,” the teacher asked: “Now someone will tell me what the heck a quadrilateral is? ’Cause I hadn’t heard that word yet.” By requesting that students define objects that they introduced, the teacher encouraged the development of concepts in a systematic way—entities referred to in their definitions also needed to be defined and understood. These teacher requests were often accompanied by messages about why certain forms of participation were important, such as the teacher’s implied note that quadrilaterals had not yet entered the common conversational ground or that he had simply not heard it because he was not part of the initial lessons about polygons.

After students had proposed initial ideas, the teacher further encouraged coordination between concepts and aspects of practice by sometimes requesting that students participate in another aspect of practice, reasoning about systematic relations. He did so by asking definitional questions that served to expand the mathematical system. For example, as they continued to reason about Lavona’s proposed definition, one group of students, Mona, Kate, and Adeena, called out that “a circle wouldn’t be a polygon cause a circle doesn’t have sides.” In constructing this definitional argument, the students introduced a new object, circle, to the discussion. Rather than accepting the students’ proposition, the
teacher revoiced their comment as a question: “Okay so QUESTION. Circle is? A polygon?” This allowed the teacher to request student participation in reasoning about systematic relations.

The teacher’s requests for students’ participation in other aspects of practice also played an important role in contributing to concept development. For example, he drew a long, thin rectangle on the board and invited students to evaluate the example as a regular polygon. When some students were unsure, he requested that students construct a definitional argument by reasoning about the example: “How could you convince them (those students who do not agree) that it (the rectangle) is NOT regular?”

Through these interactions, the teacher continuously modeled participation in asking definitional questions and, in particular, asking questions about the meaning of objects related to what students were discussing. Such questions were important because they served to expand the mathematical system that students were coconstructing. As we illustrate in a later excerpt, this modeling had possible implications for how students began to appropriate participation in aspects of practice in ways that were important for supporting the development of the mathematical system.

The Role of Monsters in Expanding the Mathematical System

On Day 4, when students revisited their definition of polygon, they proposed many of the same ideas as on Day 1 and concluded that a polygon should be defined as having sides and angles. Up until now, the teacher’s requests for students to participate in aspects of practice and his use of definitional questions expanded the mathematical system being considered and motivated the introduction of new ideas and properties. However, at this point, the teacher shifted tactics and introduced monsters (Lakatos, 1976) that problematized students’ definitions. The teacher created each monster to differ from the students’ concept images of the object, and the feature that caused the monster to be undesirable was what students needed to add to their definition to discount the monster.

For example, the teacher drew three connected but not closed lines (a zigzag appearance) and said, “I want to know what makes something a polygon. I know it has sides and it has angles SO . . . this then is a polygon right? . . . (labels the figure as students talk) side one, side two, side three, angle one, angle two.” In arguing for his monster, the teacher was careful to relate it to the students’ definition, thus motivating the need for revision. Because the zigzag consisted of straight sides but was not closed, it prompted students to describe the need for the property of closure. By stating, “This then is a polygon right?” the teacher invited students to participate in evaluating his example. Students, with much emotion, protested and argued that the figure was not a polygon, until one student, Owen, stated that “it has to be CO::nnected.” The teacher added this revision to their definition on the board and then suggested alternate language they could use to express the same idea: “Sometimes we say that it’s closed. Meaning that it has an inside, and an outside.” Figure 3 illustrates how the introduction of the zigzag monster prompted students’ participation in the aspects of practice of evaluating examples, describing properties,
and revising definitions while also contributing the new property of “connected,” or (as the teacher then revoiced) “closure,” to their mathematical system.

Figure 3. Proposing examples that create contest (i.e., monsters). The figure illustrates how a monster prompted student participation in aspects of practice in ways that also helped them elaborate the system components. Aspects of practice in which members of the class participated are indicated in the gray rectangles. Ovals represent classes of objects, properties, examples or nonexamples mentioned by members of the class. Lines represent relations (both correct and incorrect) between objects, properties, and examples or nonexamples, where a relation is defined as inclusion or exclusion between two of the entities (e.g., a polygon “has to be” connected or the zigzag monster “is” a polygon). Exclusion is denoted by an “x” on the line. Labeled edges indicate the nature of the relation between the two properties (e.g., “same” is used to show that the students described polygons as having the same sides and same angles). Bold and underlined transcript is used to indicate mention of the objects, properties, or relations introduced in that frame.

Later, after the idea of “side” arose, students proposed definitions of “side.” One student, Diego, said, “I think a side is a line that’s connected to another line.” The teacher drew an example in order to provoke contest. This time, he drew a closed figure with one curved line (see Figure 4) and noted, “I had a line, and then I connected it and then I connected it again. Do we want to call this thing (points to the curved side) a line?” As before, in his argument, he attended to the student’s definition, motivating the need for revision. Again, his question invited students to evaluate his example, and Lavona noted, “It has to be STRAIGHT.”
When introducing monsters, the teacher not only continued to model the asking of definitional questions, he also modeled other aspects of practice, especially how one might construct and evaluate examples and describe properties in service of a definitional argument. In what follows, we illustrate students’ appropriation of these aspects of practice in Days 4 and 6.

Students Taking on Authority for Expanding the Mathematical System

In the definitional episodes analyzed in Days 4 and 6, students began to ask definitional questions and construct examples similar to the teacher’s and, in doing so, played an important role in promoting expansion of their mathematical system. Their monsters and questions played important roles in introducing and encouraging participation in other aspects of practice, through which students investigated new concepts. These aspects of practice prompted students to engage in other aspects of practice, most prominently establishing and reasoning about systematic relations, describing properties, constructing and evaluating examples, constructing explanations or arguments, and proposing definitions. Within a short amount of time, the concepts of “side,” “straight,” and “2D” surfaced and were discussed, leading to later discussions of additional properties and relations.

For example, in Day 4, after the zigzag monster prompted revision of their definition of a polygon, the teacher returned to a question Kira had asked earlier: “If we take this definition, can there be a polygon with two sides?” Kate suggested that as long as the two sides were connected, it was possible, and then suggested an example of an oval. Another student gestured an oval to illustrate what she thought Kate meant, and the teacher drew her interpretation on the board, making it accessible to others in the class (see Figure 5). Much like the zigzag monster, the drawn oval caused many in the class to protest. Amidst the disagreement, Mona, whispered to her tablemates, “What’s a side?” The other girls chimed in, with Adeena asking loudly to their peers, “What’s a side, people?” This definitional question resembled those the teacher had been modeling because it served to elaborate on the mathematical system the class was exploring.

What we find noteworthy about this appropriation of aspects of practice is that Mona and her peers understood not only what type of question to ask but also when
Figure 5. Student example of a polygon with two sides, drawn by the teacher.

to ask it. Similarly, later in the same day after the idea of “straight” had arisen, students had continued to discuss sides and whether they needed to be straight. When they reached an impasse, the teacher noted, “But I don’t know what I mean by side yet. I heard the word STRAIGHT.” Vern then followed with the definitional question: “What does straight mean?” His inflection in asking the question suggested that he was unsure about the appropriateness of the question. The teacher affirmed his choice by commenting, “Yes, what does straight mean?”

At the same time though, students were still constructing normative understandings for when such questions are appropriate and the purpose they serve. For example, in the initial discussion, Lavona had asked, “What makes it [a polygon] regular?” even though a few minutes earlier the class had defined regular and written it on the board. The teacher helped establish which questions were important by writing new questions down and dedicating more time to discussing them. Such supports seemed to help because in later discussions, it appeared that some of the students had developed dispositions toward posing definitional questions when the intended referents of an object were not clear. For example, on Day 8 during a presentation to the class about triangle angle sums, Cordell started to doubt whether what he had drawn on the board was a triangle: “This, it has uh these two are the same size but these (points to the third side) ain’t and I think a triangle supposed to have congruent sides.” The teacher then asked if his figure was a triangle. Cordell responded, “I don’t know what it’s called (looks at class). What is this called?”

Further appropriation of aspects of practice occurred at the beginning of Day 6 of instruction after the teacher returned to the question of defining polygon once more. Whereas earlier the teacher’s examples had largely been the source of contest and revision of definitions, now the students’ constructed examples motivated reconsideration of the ideas they had been exploring. One student, Mataya, read from her notebook: “It is a closed figure that has angles and sides.” The teacher wrote the definition on the board and then returned to a definitional question about economy he had posed the day before: “Can you make any closed figure with sides that does NOT have angles?” Two students, Ned and Kira, suggested that they could, and the teacher asked them to draw an example. By insisting that they draw and then describe their drawings, he requested that they participate in the aspect of definitional practice of constructing and evaluating.
examples of objects being defined. Ned drew a football-shaped figure (see Figure 6) and, when prompted by the teacher to “help us understand how you’re thinking,” defended his example. His example, however, was met with disagreement from Kate.

\[ N: \text{(points to the two “sides”) Two sides. (points to the vertices) No angles.} \]
\[ T: \text{[Two sides, no angles.]} \]
\[ N: \text{[They can’t be angles] cause an angle has to be a straight line, [two straight lines] make an angle (uses his hand to show two potential straight lines, see Figure 6).} \]
\[ T: \text{[Has to have] An angle has the intersection of two, lines? Two straight lines? Okay. Does anyone have a counterargument for Ned? Kate. (Kate looks confused) Well, can you argue with Ned. Do you, do you agree with Ned or not?} \]
\[ K: \text{Um I don’t cause that’s not a polygon.} \]
\[ T: \text{Okay.} \]
\[ K: \text{And Mataya forgot to say [that it has to have straight lines.]} \]
\[ T: \text{[I think you need to say that to Ned] though.} \]
\[ K: \text{(turns to Ned) That’s not a polygon.} \]
\[ N: \text{Did he say it had to be a polygon?} \]
\[ SS: \text{Yeah.}^1 \]
\[ K: \text{’Cause based on, based on Mataya’s [um thing.]} \]

![Figure 6. Ned’s example of a polygon with sides but no angles. Here his gestures are meant to show that the angles are not made up of straight lines.](image)

In constructing his argument to defend his example, Ned described the properties of the figure and then appealed to the definition of angle to make his case (“they can’t be angles, cause an angle has to be two straight lines, two straight lines make an angle”), suggesting that he considered both to be important forms of evidence. His appeal to the definition resembled the teacher’s earlier arguments about the zigzag and curved triangle. Ned’s football construction also resembled

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1 SS stands for students (plural).
the teacher’s earlier monsters because it was counter to what students considered a polygon to look like. Thus, his construction also provoked contest and prompted Kate to engage in constructing a definitional argument. In her counterargument, she too appealed to definitional properties by noting that Mataya should have included the notion of straight in their definition. The teacher then pointed out that the difference in Kate’s and Ned’s thinking was what they considered a “side” to be. Ned furthered this point by posing the definitional question “What did we say a side is?” This question, like others before, once again directed the conversation toward considering the definition of side. After some discussion, Kate reiterated her earlier argument that the “lines have to be straight,” and others also agreed. The teacher articulated another rule for defining, namely, that once they agreed upon the definition, they would need to stick with it: “Once we say this, then this is what we mean. A side is a line. And we said it usually has a beginning and an end point. A line segment that is STRAIGHT.”

Ned’s example (monster) and his question about sides encouraged the class to revisit their definition of side and enforce the notion that the definition implied straightness. In turn, this provided an opportunity for the class to revisit their ideas about straightness that they had extensively constructed on Day 4. Later, Kira’s example (a marker cap), also a monster, prompted the class to add the property of “2D” to their definition. Moreover, because of students’ discussion of the relation of polygon and circle during the previous class, they more readily rejected a circle as a polygon. This consensual idea served as a resource for evaluating Kira’s example.

Kate’s and Ned’s contributions also illustrated an awareness of the significance of describing properties of agreed upon definitions when evaluating examples, something the teacher had been consistently modeling and articulating through messages about expectations for participation in defining. Although Kate had used definitions to justify inclusion and exclusion of examples and other definitions even as early as the first day, here she explicitly referred to their communal definition by noting that “Mataya forgot to say” and “cause based on, based on Mataya’s um thing.” Although subtle, this reference to their definition resembled the teacher’s previous talk (e.g., “according to our definition”). In what follows, we illustrate students’ further development of these dispositions in Day 26.

**Student Dispositions Toward Defining**

On Day 26, in order to encourage the students to think about the relations among a triangle’s properties, the teacher asked the students to construct economical definitions for triangle in their table groups to then share in the whole group. Two changes were noteworthy of students’ participation in defining triangles. First, on Day 26, the conversation, compared to that on Day 1, was more focused on properties and relations, most of which were student initiated. All students’ definitions included the properties of “three sides” and “closed,” and most included straightness. Some definitions included properties from their recent investigations, including angles sums and (no) diagonals. In contrast to their definitions of polygon, their triangle definitions were created with little scaffolding and within
a much shorter time frame. Moreover, there were fewer deviations on Day 26; that is, the conversation was more focused on definition construction. On Day 1, the teacher often had to remind students of their goal of creating a definition of polygon and, later, a definition of regular. When asked to define polygon, students on that initial day listed many examples of polygons (e.g., octagon, quadrilateral) without contributing directly to the creation of a definition.

The students’ attendance to properties and relations on Day 26 suggests that they had developed an inclination to seeing definitions as a means of distinguishing a class of objects from others. Moreover, this propensity was not limited to a few students. All the groups of students constructed definitions, and students who had been new to the class were important contributors during the discussion, especially by encouraging the use of agreed upon terms (e.g., congruent) in definitions. In fact, during both days, more than half of the students contributed to the development of the mathematical system (67% on Day 1 and 78% on Day 26). The transitional episodes between Day 6 and 26 generally showed students’ increased propensity to describing properties, although not as pronounced as Day 26. For example, on Day 8, when asked about the definition of a triangle, Lavona said that a triangle “has straight sides down and doesn’t slant or like anything and it’s got the two sides that are going down are equal.” Thus, although Lavona noted the triangle’s straight sides, part of her definition was based more on the shape’s appearance. The teacher pressed on her description by asking, “So what do you mean straight sides down? I’m not sure I understand.” The teacher continued to support students’ attendance to properties in similar ways in other days leading up to Day 26.

Also noteworthy was that in their small groups, students began to take on the role of the teacher in orchestrating participation in defining. Kate, Mona, and Adeena’s group almost immediately constructed a definition of “only three sides, only three angles and closed.” When Mona suggested that sides needed to be straight, Kate reminded her that “we already knew sides were straight . . . that’s the definition of side.” This interaction resembles that from Day 6 of instruction when students had discussed Ned’s football example. In that instance, Kate had suggested that their definition of polygon needed to include straight, and the teacher had noted that once they establish that a side means straight, then they no longer need to specify so. Later when Adeena proposed the same idea, Mona reiterated Kate’s message. At another juncture in the Day 26 interaction, Adeena proposed that they needed to specify that the sides and angles be “equal.” Mona quickly countered this proposal, stating, “No that’s for a regular. Does everything have to be regular? No::: I don’t think.”

Similar negotiations occurred in another group. Diyari started by suggesting that a triangle was “a 3-sided figure.” Jomerd elaborated with “a 3-sided, closed” figure, and Diyari, in turn, suggested “polygon.” As they argued over whether to use the words polygon or figure, Cordell continued to write a definition. When he shared his version, “a triangle is a three-sided figure that has a turn angle of 120,” Diyari immediately presented a counterexample, a monster, in a manner similar to how the teacher had in the initial days.
D: Nu-uh, not all of them do. This is a triangle (draws something). That’s a triangle.
C: That’s not a regular triangle.
D: But you just wrote a triangle (points to Cordell’s notebook). You didn’t write a regular triangle.
C: (writes something in his notebook) A regular triangle.

In the interaction between the boys, Diyari, taking on the role the teacher had earlier modeled, prompted Cordell to revise his definition. He did so by presenting a counterexample and by attending to Cordell’s definition by pointing to his notebook and noting, “But you just wrote a triangle.” These small-group interactions suggest that students were participating in aspects of practice that had by now become more stable and needed less or no support from the teacher.

Students’ participation during the ensuing whole-group discussion further suggested a greater appeal to properties and to agreed upon definitions. For example, the teacher began the discussion by reading Kate, Mona, and Adeena’s definition: “three sides, three angles only and it is closed.” He then asked a definitional question that encouraged the students to consider whether their definition was inclusive enough: “Can anyone think of something that their definition, it wouldn’t work for it? Or something that is not triangular but their definition would seem to fit it?” Several students raised their hands, and the teacher called on Vern, who suggested “straight sides.” The three girls immediately protested, arguing that their definition of sides implied the notion of straightness.

A: But we already said sides.
T: [So this assumes that the]
M: [That’s the [definition of sides.]]
K: [definition of sides.]

Here, Kate and her tablemates negotiated with Vern about whether or not to include straight, implying that there was no need to based upon their definition and previous classroom consensus. At the same time, however, Vern’s contribution was still valuable because he not only described the properties of triangle but also reasoned about the systematic relations between its properties and subproperties, aspects of practice that the students had spent extensive time developing within the first few classes of the school year.

Although the students took on greater agency, the teacher still played an important role in supporting the codevelopment of concepts and aspects of practice. As in the early classes, he initiated defining by requesting that students propose definitions. However, this time, the focus was more on economical definitions with the pedagogical intention of promoting closer inspection of relations among properties. Although he had asked students definitional questions about
economy during Days 5 and 6, this time he started with a more open-ended request and then followed up during the discussion with particular probes (e.g., “Do they need to say closed if they say polygon?”). Moreover, the teacher again articulated messages about participating in defining. Although some messages were similar to earlier ones (e.g., “Write me a definition of triangle so that, so that we can know for sure . . . that, what we’re looking at is a triangle.”), others differed given the greater focus on economy. For instance, the teacher labeled or coded (Goodwin, 1994) students’ definitions using descriptors such as *slim*, *sparse*, *works*, and *good enough* to highlight the degree to which they were necessary or sufficient. Thus, as the students explored new mathematical ideas of relations among properties, the teacher’s participation in aspects of practice shifted accordingly.

**Discussion**

In this article, our goal was to investigate how aspects of the practice of defining and concept development were coconstituted over time, with a focus on the forms of interaction that facilitated this coordination. We identified five types of interactions that helped to explain how aspects of definitional practice and mathematical concepts were coconstituted. The first type of interaction, requesting that members of the class participate in various aspects of practice, was important for providing students opportunities to immediately participate in aspects of definitional practice. In particular, the teacher most often requested that students participate in proposing potential definitions, reasoning about systematic relations, evaluating or constructing examples, and constructing definitional arguments. In her study of the practice of generalization, Ellis (2011) noted that members of the class supported engagement in the practice through a similar action, *encouraging generalizing*, which involved encouraging students to engage in various aspects of the practice, including relating, searching, extending, and reflection. Thus, a productive avenue for initiating students into mathematical practice may be to deliberately request or encourage participation in aspects of that practice.

In turn, requests for students to participate in aspects of definitional practice provided an invitation for students to immediately contribute ideas to the mathematical system. However, our analysis suggests that in order to do so, the teacher should request participation in aspects of practice in ways that also encourage students to expand the mathematical system. For example, the teacher, especially early on, asked questions that encouraged students to propose definitions of objects or properties related to those being discussed or to reason about systematic relations. Thus, to coordinate practice with concept development, teachers should orient their requests toward expanding the mathematical system. By the end of instruction, students in our study had defined and elaborated on a network of geometric objects and relations which in turn were objects for other practices, such as posing questions and making and justifying conjectures. This ensemble of mathematical activity was grounded in definition and the negotiations that definitions inspired.
Additionally, the teacher’s modeling of engagement in aspects of practice appeared important for supporting students in learning not only how but also when to engage in defining. Initially, the teacher played a greater role in initiating interactions, but over time, students took on increasing agency and appropriated these forms of support. As the teacher modeled aspects of definitional practice, he actively engaged in defining with students rather than just asking them to participate in the aspects of practice. Although modeling has been well established as important for supporting students’ mathematical work (Lampert, 2001), here we have shown what particular interactions the teacher should model in order to support engagement in defining. For example, the teacher initially asked definitional questions that probed into the meaning of new objects or properties that students proposed and often encouraged students to resolve disagreements that arose. Likewise, students later asked similar questions in response to disagreement (e.g., “What do we mean by sides, people?”).

The teacher also acted to destabilize consensual definitions by proposing monsters, which typically generated new properties and instigated consideration of relations among properties. Appropriating this strategy, students proposed monsters as well (e.g., the two-sided polygon). Thus, by providing examples, students engaged in definitional arguments that encouraged the revision of definitions and the expansion of the mathematical system (e.g., new concepts were proposed that allowed for more necessary, property-rich definitions). Monsters often provoked explosions of contributions, suggesting that students were invested in the practice and eager to participate. Our study confirms similar findings pointing to the importance of examples in helping students make sense of mathematical concepts (e.g., Ambrose & Kenehan, 2009; Dahlberg & Houseman, 1997; Roth & Thom, 2009) and, in particular, of nonexamples (e.g., de Villiers, 1998; Lehrer & Curtis, 2000; Lin & Yang, 2002; Zaslavsky & Shir, 2005) or unfamiliar examples (e.g., Lehrer et al., 1999; Zandieh & Rasmussen, 2010) in provoking students to expand their definitions. However, we extend this work by highlighting how nonexamples in particular should be designed, given students’ starting points, and when it might be productive to present nonexamples. For example, the monsters that provoked changes in the students’ definitions all shared the following properties: (a) each differed from the students’ concept image of the object, (b) each was consistent with students’ definition at the time, and (c) the property that caused the monster to be undesirable was that which students needed to add to their definition to discount the monster.

Finally, the teacher explicitly stated expectations for how to participate in defining and purposes for constructing definitions. These included expectations that: (a) definitions should help to distinguish between objects, (b) after we agree with a definition, then we use that definition, and (c) it is important to keep track of our definitions so that we can revise them as needed. Our analysis of these expectations expands work illustrating the importance of establishing sociomathematical norms (Yackel & Cobb, 1996) but highlights norms specific to defining activity. Similar forms of teacher support for learning to participate in defining
included highlighting students’ contributions in talk and writing (e.g., writing their definitions on the board or praising agreed upon language: “THANK YOU Terrance . . . that math word says it all”) and explicitly acknowledging students as authors (e.g., “Vincent’s claim is that all polygons have the same length of sides and the same angles”). The students’ sense of authority was evidenced by their readiness to contribute definitions, examples, and counterarguments, including arguments counter to those of the teacher.

The five interactions described above are related, but we note them separately because we believe each plays an important role in supporting codevelopment of defining and mathematical concepts. In addition to what we describe above, our analysis of these interactions contributes to and extends previous work in mathematics education in three important ways. We elaborate on each contribution and the related implications in what follows.

Providing Frameworks for Decompositions of Practice

The framework of the eight aspects of definitional practice provides a useful analytic framework for investigating students’ participation in the practice of defining. One important contribution of our work is the creation of this framework that started with the synthesis of existing research about engaging students in defining. This suggests that these aspects of practice are not isolated to the particular teacher and students in our study. Such decompositions of practice are necessary to make empirical progress to sustain the call in the Common Core State Standards for Mathematics (NGA & CCSSO, 2010) to initiate students in forms of mathematical practice. The Common Core lays out a progression of concepts but does not provide a similar progression for practices. If the practices are to be implemented in classrooms, more work is needed to understand the development of the practices in relation to existing trajectories of concepts. However, in order to create developmental trajectories, descriptions of the constituents of practice are necessary. In our study, the aspects of defining framework allowed us to infer how members of the class positioned themselves during classroom interactions, to note how their participation in particular aspects of practice contributed to the development of a mathematical system, and to examine how those forms of interaction shifted over time. For example, using our framework, we noticed how the teacher’s requests for participation involved engagement in particular aspects of practice, which in turn resulted in student contributions to the mathematical system.

Although some have provided decompositions of practice related to specific mathematical topics (e.g., Bowers, Cobb, & McClain, 1999; Moschkovich, 2004; Stephan & Rasmussen, 2002), here we suggest a decomposition of a general epistemic practice that spans multiple topics. Similar to work by Ellis (2011) and Jurow (2004), we have sought to distinguish between the product of epistemic practice (e.g., the definition) and the actions involved in engaging in constructing that product (e.g., the aspects of defining). Moreover, many of the episodes of definition revision involved increased object scope, a form of generalization. We argue that further investigations of mathematical practice might include similar
decompositions of other practices in order to make sense of how these forms of activity are realized in the course of interactions among students and teachers.

Aspects of definitional practice also hold promise as a professional development tool. Jacobs, Lamb, and Philipp (2010) described the importance of developing teachers’ “professional noticing” of student thinking, which first involves characterizing how students think and then using those characterizations to inform teaching. Likewise, in order to apprentice students into mathematical practices, teachers should also develop professional noticing of those practices. However, doing so requires an understanding of what the practice entails. Thus, we envision that frameworks such as ours may be used to help support teachers in developing a view of the forms of participation they hope to support their students in developing.

Providing a View of Development

Most previous work in mathematical defining typically considered relatively brief spans of time (e.g., Ambrose & Kenehan, 2009; de Villiers, 1998; Herbst, 2005; Herbst et al., 2005; Larsen & Zandieh, 2005; Lehrer et al., 1989; Lehrer et al., 1999; Lehrer & Curtis, 2000; Leikin & Winicki-Landman, 2001; Lin & Yang, 2002; Ouvrier-Buffet, 2006). Studies that described longer durations of time either focused mainly on changes in conceptions (Keiser, 2000), were limited to small groups of students (Borasi, 1992; Zaslavsky & Shir, 2005), or looked at the development of students with existing histories of engagement in defining and other mathematical practices (Zandieh & Rasmussen, 2010). In contrast, here we illustrate how the practice of defining developed for an entire class of students during a prolonged period of time. Our focus on development involved tracing how opportunities to participate in aspects of mathematical definition were accomplished in interactions between students and teachers and how these interactions showed evidence of student appropriation of aspects of practice originally initiated by teachers. Over time, students appeared more disposed to consider definition as an important way of establishing common mathematical ground, to hold members of the class accountable for violations of the common ground, and to generate forms of definition that relied increasingly on concise descriptions of properties and relations among these properties. The latter are forms of thinking about shape and form that are typically challenging for students of this age (van Hiele, 1986). By inviting students to participate in constructing and evaluating definitions, students developed inclinations toward considering and elaborating properties of objects that afforded communication and grounds for agreement. By considering economic definitions, students probed more deeply into relations among properties. As students presented multiple ideas and negotiated the meaning of these ideas, they generated multifaceted notions of geometric objects, their properties, and relations among these properties (Keiser, 2000). Hence, learning to participate in aspects of definitional practice and learning about the mathematics of space co-originated. Further, as we have mentioned, defining created the common ground for the development of related practices, such as posing questions worthy of investigation (for more information about this work, see Lehrer et al., 2013).
Providing an Analysis of Concepts in Practice

We began this article by pointing to the recent movement in mathematics education toward a view of concepts as arising from participation in mathematical practices. Our work suggests that in order to cultivate classroom environments in which students participate in disciplinary practices that contribute to the development of concepts, teachers must expand their view of a “mathematical horizon” (Ball, 1993, p. 376) to include prospective development of mathematical practices that are interwoven with pathways of conceptual development. Yet, mathematical practices are apt to be complex and in need of productive decompositions to facilitate pedagogy. Thus, current notions of mathematical knowledge for teaching (e.g., Hill, Ball, & Schilling, 2008) and how we support teachers in developing that knowledge may need to be further developed to take into account the codevelopment of mathematical practice and concepts. For example, teachers must be able to know when and how to respond to students’ definitions with monsters that productively expand the space of mathematical conversation. Our work provides a first step in advancing this knowledge by providing a framework for one epistemic practice. Moreover, our description of five productive forms of pedagogical support of student participation in this practice suggests a structure for describing the choices teachers need to make in order to support students’ engagement in this practice.

References


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Submitted December 29, 2013
Accepted May 26, 2014